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International Journal of Solids and Structures xxx (2005) xxx–xxx

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

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Bending of beams on three-parameter elastic foundation

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Received 31 August 2004; received in revised form 6 March 2005

Abstract

The bending of a Timoshenko beam resting on a Kerr-type three-parameter elastic foundation is introduced, its governing differential equations are formulated and analytically solved, and the solutions are discussed and applied to particular problems. Parametric analyses of elastically supported beams of infinite and finite length are carried out and comparisons are made between one, two or three-parameter foundation models and more accurate 2D finite element models. In order to estimate the necessary soil parameters, an analytical procedure based on the modified Vlasov model is proposed. The presented solutions and applications show the superiority of the Kerr-type foundation model compared to one or two-parameter models.

Keywords: Beams on elastic foundation; Two-parameter elastic foundation; Three-parameter elastic foundation; Timoshenko beam; Finite element method

1. Introduction

The problem of a beam resting on elastic foundation is very often encountered in the analysis of building, geotechnical, highway, and railroad structures. Its solution demands the modeling of (a) the mechanical behavior of the beam, (b) the mechanical behavior of the soil as elastic subgrade and (c) the form of interaction between the beam and the soil.

As far as the beam is concerned, most engineering analyses are based on the classical Bernoulli–Euler theory, in which straight lines or planes normal to the neutral beam axis remain straight and normal after deformation. This theory thus neglects the effect of transverse shear deformations, a condition that holds

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doi:10.1016/j.ijsolstr.2005.03.033

only in the case of slender beams. It is well-known that the error in shear force and moment distribution can become significant in the case of foundation beams with small length-to-depth ratio subjected to closely spaced discrete column loads, as well as in the case of flanged beams and beams with sandwich-like cross-section. To confront this problem, the well-known Timoshenko beam model, in which the effect of transverse shear deflections is considered, can be used.

While fairly realistic and efficient models of the material properties and the mechanical behavior of the beam can be established by using the Timoshenko or even the Bernoulli–Euler theory, the characteristics that represent the mechanical behavior of the subsoil and its interaction with the beam resting on it are difficult to model. Assuming a linear elastic, homogeneous and isotropic behavior of the soil, two major classes of soil models can be identified in the literature: (i) the continuous medium models, and (ii) the so-called "mechanical" models.

The continuous medium models, based on the fundamental hypothesis of an elastic semi-infinite space, are more accurate, but it is difficult to obtain an exact analytical solution even after introducing simplifying assumptions (Selvadurai, 1979). Of course, on the basis of practical considerations, the effective foundation area can be restricted to finite dimensions, and the finite element technique can be applied to obtain numerical results. Furthermore, potentially existing symmetries in the superstructure can be utilized to reduce the initial 3D problem to a 2D one. Although such finite element models guarantee a rather precise calculation of stresses and deformations, they require significant computer capacity and processing time and, more importantly, thorough knowledge and sound judgment on the part of the engineer. On the other hand, mechanical models are clearly less precise, but conceptually simple and easier to use. The oldest, most famous and most frequently used mechanical model is the one devised by Winkler (1867), in which the beam-supporting soil is modeled as a series of closely spaced, mutually independent, linear elastic vertical springs which, evidently, provide resistance in direct proportion to the deflection of the beam. In the Winkler model, the properties of the soil are described only by the parameter k, which represents the stiffness of the vertical spring. Thanks to its simple mathematic formulation, this one-parameter model can be easily employed in a variety of problems (Hetenyi, 1946) and gives satisfactory results in many practical situations. However, it is considered as a rather crude approximation of the true mechanical behavior of the soil material, mainly due to its inability to take into account the continuity or cohesion of the soil. This limitation, i.e., the assumption that there is no interaction between adjacent springs, also results in overlooking the influence of the soil on either side of the beam. To overcome this weakness, several two-parameter elastic foundation models have been suggested (Filonenko-Borodich, 1940; Pasternak, 1954; Vlasov and Leontiev, 1966). In these models, the first parameter represents the stiffness of the vertical spring, as in the Winkler model, whereas the second parameter is introduced to account for the coupling effect of the linear elastic springs. It is worth mentioning that the interaction enabled by this second parameter also allows the consideration of the influence of the soil on either side of the beam. Despite the introduction of a second parameter, the mathematical formulation of the problem and the corresponding analytical solutions remain relatively simple (Selvadurai, 1979). Thus, two-parameter models are less restrictive than the Winkler model but not as complicated as the elastic continuum model. It is interesting to note that, in developing foundation models, two major procedures are normally used. The first starts with the soil as a semi-infinite continuum and then introduce simplifying assumptions with respect to displacements or/and stresses in order to proceed to analytically or numerically solvable relations. The second starts with the simplest mechanical model, i.e. the Winkler model and, in order to bring it closer to reality, assume various kinds of interaction between the independent springs. Both procedures can be used in order to developed more sophisticated models comprising three independent parameters for the description of the soil behaviour. These three-parameter models constitute a generalization of two-parameter models, the third parameter being used to make them more realistic and effective. This category includes the models developed by Kerr, Hetenyi and Reissner (Kerr, 1965). One of the basic features of the three-parameter models is the flexibility and convenience that they offer in the determination of the level of "continuity" of the vertical displacements at the boundaries between the loaded and the unloaded surfaces of the soil (Hetenyi, 1950). This feature renders them capable of distributing stresses correctly, whether the soil is cohesive or non-cohesive. Among all three-parameter models, the Kerr model is of particular interest. It represents a generalization of the two-parameter Pasternak model for which a series of solutions and applications are already available. The Reissner model (1967), which was studied by Horvath (1983a,b, 1993), is also worthy of consideration. His study led to far-reaching conclusions that constitute proof of the superiority of the tree-parameter model over the rest of the mechanical models.

In the present paper, the governing equations for the bending of a Timoshenko beam on a Kerr-type three-parameter elastic foundation are developed and solved. As no solution can be applied in the real world without realistic estimates for the soil parameters involved, an analytical process of estimating the three parameters of the Kerr model is also presented. The estimation of the first two parameters is based on the modified Vlasov model (Vallabhan and Das, 1988, 1991a,b), while a parametric investigation, which forms an integral part of the aforementioned process, leads to the estimation of the third parameter. This investigation is carried out through comparisons of the results obtained from the analysis using the three-parameter model with results from the analysis using 2D finite element models. Apart from presenting a working solution for a three-parameter modeling procedure for Timoshenko beams resting on elastic foundation, the main objective of the present study is to clarify the particular use of the Kerr model and to demonstrate that it is more precise compared to one or two-parameter models, while remaining relatively simple.

2. Formulation of the differential equations

The Kerr model (1965) was introduced as an attempt to produce a generalization of the two-parameter Pasternak model (1954). It consists of two linear elastic spring layers of constants c [kN/m^3] and k [kN/m^3], respectively, interconnected by a unit thickness shear layer of constant G [kN/m] (Fig. 1). The governing

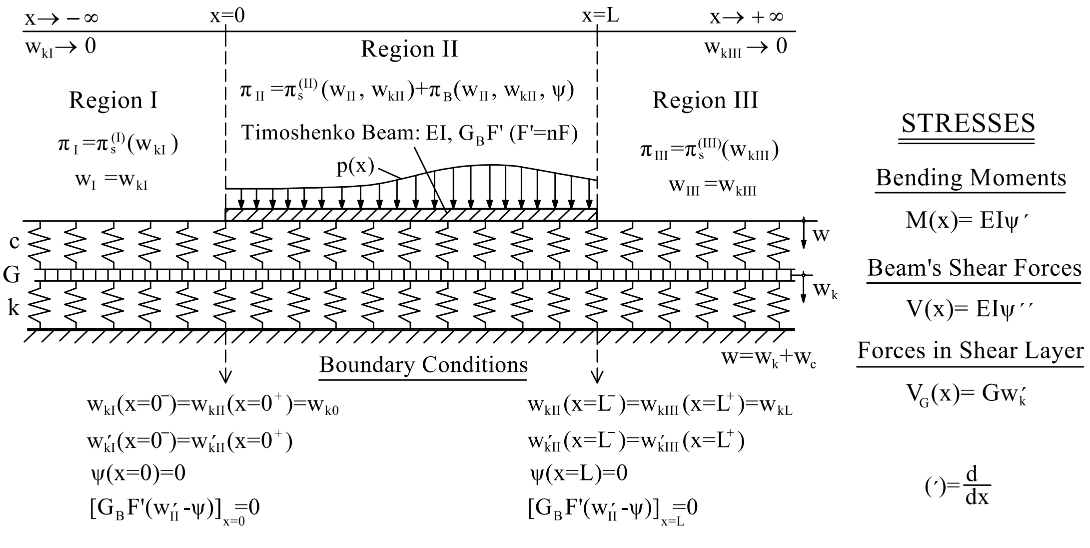


Fig. 1. Mechanical model, boundary conditions and basic stress–strain relationships for a Timoshenko beam resting on Kerr type elastic subgrade.

differential equations describing the behavior of this mechanical model are based on the following two basic assumptions:

- The beam resting on the Kerr-type elastic subgrade is a Timoshenko beam with constant cross-section, in which the effect of the shear on the curvature is taken into account.
- Plain strain conditions, which allow the consideration of a foundation strip of infinite length and finite width b, are valid.

In Fig. 1, the basic stress–strain relationships and the boundary conditions for a Timoshenko beam resting on Kerr-type elastic subgrade are given. In order to derive the governing differential equations, the principle of stationary potential energy is used. The potential energy function of the soil-beam system is given by

$$\pi = [\pi_{\mathrm{B}}(w_{\mathrm{II}}, w_{k\mathrm{II}}, \psi) + \pi_{\mathrm{S}}^{(\mathrm{II})}(w_{\mathrm{II}}, w_{k\mathrm{II}})] + \pi_{\mathrm{S}}^{(\mathrm{I})}(w_{k\mathrm{I}}) + \pi_{\mathrm{S}}^{(\mathrm{III})}(w_{k\mathrm{III}}) \Rightarrow \pi = \pi_{\mathrm{I}} + \pi_{\mathrm{II}} + \pi_{\mathrm{III}} \tag{1}$$

As indicated by Eq. (1), the potential energy consists of three terms, each of them corresponding to the deformation energy of the three regions that comprise the soil-beam system (Fig. 1). These terms are:

$$\text{Region I}: \quad \pi_{\mathrm{I}} = \frac{b}{2}\int_{-\infty}^{0} \overline{K}[w_{k\mathrm{I}}]^2\,\mathrm{d}x + \frac{b}{2}\int_{-\infty}^{0} \overline{G}\left(\frac{\mathrm{d}w_{k\mathrm{I}}}{\mathrm{d}x}\right)^2 \mathrm{d}x \tag{2a}$$

$$\begin{aligned}\text{Region II}: \quad \pi_{\mathrm{II}} = {} & \frac{1}{2}\int_{0}^{L} EI\left(\frac{\mathrm{d}\psi}{\mathrm{d}x}\right)^2 \mathrm{d}x + \frac{1}{2}\int_{0}^{L} \Phi\left(\frac{\mathrm{d}w_{\mathrm{II}}}{\mathrm{d}x} - \psi\right)^2 \mathrm{d}x + \frac{b}{2}\int_{0}^{L} \overline{C}(w_{\mathrm{II}} - w_{k\mathrm{II}})^2\,\mathrm{d}x \\ & + \frac{b}{2}\int_{0}^{L} \overline{K}[w_{k\mathrm{II}}]^2\,\mathrm{d}x + \frac{b}{2}\int_{0}^{L} \overline{G}\left(\frac{\mathrm{d}w_{k\mathrm{II}}}{\mathrm{d}x}\right)^2 \mathrm{d}x - \int_{0}^{L} p(x) w_{\mathrm{II}}\,\mathrm{d}x\end{aligned} \tag{2b}$$

$$\text{Region III}: \quad \pi_{\mathrm{III}} = \frac{b}{2}\int_{L}^{+\infty} \overline{K}[w_{k\mathrm{III}}]^2\,\mathrm{d}x + \frac{b}{2}\int_{L}^{+\infty} \overline{G}\left(\frac{\mathrm{d}w_{k\mathrm{III}}}{\mathrm{d}x}\right)^2 \mathrm{d}x \tag{2c}$$

In each of the above relations, w represents the total vertical displacement of the axis of the beam, w_c represents the component of w caused by the deformation of the upper spring layer and w_k represents the component of w caused by the deformation of the lower spring layer, respectively. In Eq. (2b), ψ is the bending rotation of the cross-section, $(\mathrm{d}w/\mathrm{d}x - \psi) = \beta$ is the shear rotation of the cross-section, G_{B} is the shear modulus of the beam material, F' is the effective shear area of the cross-section ($F' = nF$, F is the cross-section area, and n the shear factor) and $\Phi = G_{\mathrm{B}}F'$. Finally, b is the width of the strip of elastic subgrade, which coincides with the width of the beam.

According to the principle of stationary total potential energy, the condition of equilibrium of the system requires:

$$\pi = \text{stat.} \Rightarrow \pi = \pi_{\mathrm{I}} + \pi_{\mathrm{II}} + \pi_{\mathrm{III}} = \text{stat.} \tag{3}$$

For π to become stationary, $\delta\pi = 0$ is a necessary condition, i.e. the first variation of π must vanish:

$$\delta\pi = 0 \Rightarrow \delta\pi = \delta\pi_{\mathrm{I}} + \delta\pi_{\mathrm{II}} + \delta\pi_{\mathrm{III}} = 0 \tag{4}$$

Performing the variations of Eqs. (2a)–(2c) and integrating by parts, where necessary, the following relations are obtained:

$$\text{Region I}: \quad \delta\pi_{\mathrm{I}} = \int_{-\infty}^{0} \left[k w_{k\mathrm{I}} - G\left(\frac{\mathrm{d}^2 w_{k\mathrm{I}}}{\mathrm{d}x^2}\right)\right]\delta w_{k\mathrm{I}}\,\mathrm{d}x + \left[G\left(\frac{\mathrm{d}w_{k\mathrm{I}}}{\mathrm{d}x}\right)\delta w_{k\mathrm{I}}\right]_{-\infty}^{0} \tag{5a}$$

$$\text{Region II:}\quad \delta\pi_{\mathrm{II}} = \left[EI\left(\frac{\mathrm{d}\psi}{\mathrm{d}x}\right)\delta\psi\right]_0^L + \int_0^L \left[-EI\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \Phi\left(\psi - \frac{\mathrm{d}w_{\mathrm{II}}}{\mathrm{d}x}\right)\right]\delta\psi\,\mathrm{d}x + \left[\Phi\left(\frac{\mathrm{d}w_{\mathrm{II}}}{\mathrm{d}x} - \psi\right)\delta w_{\mathrm{II}}\right]_0^L$$
$$+ \int_0^L \left[c(w_{k\mathrm{II}} - w_{\mathrm{II}}) + kw_{k\mathrm{II}} - G\frac{\mathrm{d}^2 w_{k\mathrm{II}}}{\mathrm{d}x^2}\right]\delta w_{k\mathrm{II}}\,\mathrm{d}x + \left[G\left(\frac{\mathrm{d}w_{k\mathrm{II}}}{\mathrm{d}x}\right)\delta w_{k\mathrm{II}}\right]_0^L$$
$$+ \int_0^L \left[c(w_{\mathrm{II}} - w_{k\mathrm{II}}) - \Phi\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}w_{\mathrm{II}}}{\mathrm{d}x} - \psi\right) - p(x)\right]\delta w_{\mathrm{II}}\,\mathrm{d}x \tag{5b}$$

$$\text{Region III:}\quad \delta\pi_{\mathrm{III}} = \int_L^{+\infty} \left[kw_{k\mathrm{III}} - G\left(\frac{\mathrm{d}^2 w_{k\mathrm{III}}}{\mathrm{d}x^2}\right)\right]\delta w_{k\mathrm{III}}\,\mathrm{d}x + \left[G\left(\frac{\mathrm{d}w_{k\mathrm{III}}}{\mathrm{d}x}\right)\delta w_{k\mathrm{III}}\right]_L^{+\infty} \tag{5c}$$

where $c = \overline{C}b$ (kN/m^2), $k = \overline{K}b$ (kN/m^2), and $G = \overline{G}b$ (kN).

The differential equations of equilibrium for the three regions which constitute the soil-beam system, as well as the respective boundary conditions at $x = 0$, $x = L$ and $x \to \pm\infty$ are obtained from the requirement $\delta\pi = \delta\pi_{\mathrm{I}} + \delta\pi_{\mathrm{II}} + \delta\pi_{\mathrm{III}} = 0$.

2.1. Regions I and III

From Eqs. (5a) and (5c), the equations of the vertical displacements of the shear layer at regions I and III are obtained:

$$\text{Region I:}\quad \frac{\mathrm{d}^2 w_{k\mathrm{I}}}{\mathrm{d}x^2} - \left(\frac{k}{G}\right)w_{k\mathrm{I}} = 0 \tag{6a}$$

$$\text{Region III:}\quad \frac{\mathrm{d}^2 w_{k\mathrm{III}}}{\mathrm{d}x^2} - \left(\frac{k}{G}\right)w_{k\mathrm{III}} = 0 \tag{6b}$$

2.2. Region II

With regard to region II, which is the part of the soil surface under the beam and, therefore, the part of the soil directly loaded by the beam, the following system of three differential equations is obtained from Eq. (5b):

$$c(w_{\mathrm{II}} - w_{k\mathrm{II}}) - \Phi\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}w_{\mathrm{II}}}{\mathrm{d}x} - \psi\right) - p(x) = 0 \tag{7a}$$

$$c(w_{k\mathrm{II}} - w_{\mathrm{II}}) + kw_{k\mathrm{II}} - G\frac{\mathrm{d}^2 w_{k\mathrm{II}}}{\mathrm{d}x^2} = 0 \tag{7b}$$

$$-EI\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \Phi\left(\psi - \frac{\mathrm{d}w_{\mathrm{II}}}{\mathrm{d}x}\right) = 0 \tag{7c}$$

By combining Eqs. (7a)–(7c), the uncoupling of $w_{k\mathrm{II}}$ and ψ is achieved:

$$-\frac{EIG}{c}\frac{\mathrm{d}^6 w_{k\mathrm{II}}}{\mathrm{d}x^6} + \left[EI\left(1 + \frac{k}{c}\right) + G\frac{EI}{\Phi}\right]\frac{\mathrm{d}^4 w_{k\mathrm{II}}}{\mathrm{d}x^4} - \left[G + k\frac{EI}{\Phi}\right]\frac{\mathrm{d}^2 w_{k\mathrm{II}}}{\mathrm{d}x^2} + kw_{k\mathrm{II}} + \left(\frac{EI}{\Phi}\right)\frac{\mathrm{d}^2 p}{\mathrm{d}x^2} - p = 0 \tag{8a}$$

$$-\frac{EIG}{c}\frac{\mathrm{d}^6\psi}{\mathrm{d}x^6} + \left[EI\left(1 + \frac{k}{c}\right) + G\left(\frac{EI}{\Phi}\right)\right]\frac{\mathrm{d}^4\psi}{\mathrm{d}x^4} - \left[G + k\frac{EI}{\Phi}\right]\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + k\psi - \left(1 + \frac{k}{c}\right)\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{G}{c}\frac{\mathrm{d}^3 p}{\mathrm{d}x^3} = 0 \tag{8b}$$

Finally, using Eq. (7b), the total vertical displacement w can be computed:

$$w_{\mathrm{II}} = \left(1 + \frac{k}{c}\right) w_{k\mathrm{II}} - \left(\frac{G}{c}\right) \frac{\mathrm{d}^2 w_{k\mathrm{II}}}{\mathrm{d}x^2} \tag{9}$$

Thus, the total bending behaviour of a Timoshenko beam resting on Kerr-type elastic subgrade is governed by Eqs. (6a), (6b), (8a), (8b) and (9).

3. Solution of the differential equations

3.1. Equations of regions I and III

The general solution of Eqs. (6a) and (6b) is:

$$\text{Region I}(x \leqslant 0): \quad w_{k\mathrm{I}}(x) = D_1 \mathrm{e}^{mx} + D_2 \mathrm{e}^{-mx} (m = \sqrt{k/G}) \tag{10a}$$

$$\text{Region III}(x > L): \quad w_{k\mathrm{III}}(x) = D_3 \mathrm{e}^{m(x-L)} + D_4 \mathrm{e}^{-m(x-L)} \tag{10b}$$

The four integration constants $D_1 - D_4$ are calculated from the boundary conditions at $x = 0$, $x = L$ and $x \to \pm\infty$ (Fig. 1). It is evident that for $x \to \pm\infty$, $w_{k\mathrm{I}} = w_{k\mathrm{II}} = 0$, because the soil at regions I and III is unloaded. It must also be considered that, at the two ends of the beam ($x = 0, x = L$), the vertical displacements of the shear layer w_k are continuous (Fig. 1). Thus:

$$\text{Region I}(x \leqslant 0): \quad w_{k\mathrm{I}}(x) = w_{k0} \mathrm{e}^{mx} (x \leqslant 0) \tag{11a}$$

$$\text{Region III}(x > L): \quad w_{k\mathrm{III}}(x) = w_{kL} \mathrm{e}^{-m(x-L)} (x \geqslant L) \tag{11b}$$

Consequently, the solutions of Eqs. (10a) and (10b) depend on the unknown (at this point) vertical displacements of the shear layer at the ends of the beam, w_{k0} and w_{kL}. These can be calculated by solving the equations that govern the displacements at region II.

3.2. Equations of region II

The differential equations of region II (Eqs. (8a) and (8b)) are linear equations of the sixth order. The general form of the solutions for the homogenous part of Eqs. (8a) and (8b) is:

$$w_{k\mathrm{II}}(x) = \sum_{i=1}^{6} C_i f_i, \quad \psi(x) = \sum_{i=1}^{6} C_i^{(\mathrm{II})} f_i \tag{12}$$

C_i, $C_i^{(\mathrm{II})}$ are the integration constants and f_i are real functions, the form of which depends on the coefficients of Eqs. (8a) and (8b), as well as on a series of auxiliary parameters involved in the solution of these equations (Morfidis, 2003). As a result, the form of functions f_i depends on the values of the three parameters of the soil model and on the constants that determine the mechanical behavior of the beam (bending and shear stiffness). It can be shown (Appendix A, and Morfidis, 2003) that there exist many solution types for the homogenous part of Eqs. (8a) and (8b). However, by taking into account the usual range of values of the soil parameters (k, c, G) and of the mechanical parameters of the beams $(EI, G_{\mathrm{B}}F')$, only two types of solution (referred to as solution types 1 and 2) are of practical interest. These are composed of functions f_i, which can be found in Appendix A.

Table 1
Solution steps for the problem of infinite length beams on three-parameter elastic foundation

Step 1: *Solution of Eqs. (8a) and (8b)*
(Case 1 or 2 → Appendix A)
$w_k(x) = \sum_{i=1}^{6} C_i f_i \quad w(x) = \sum_{i=1}^{6} C_i^{(\mathrm{I})} f_i \quad \psi(x) = \sum_{i=1}^{6} C_i^{(\mathrm{II})} f_i$

Step 2: *Relation between constants $C_i,\ C_i^{(\mathrm{I})},\ C_i^{(\mathrm{II})}$*
Eq. (9) → $C_i^{(\mathrm{I})} = g_{wi}(C_i)$
Eq. (7c) → $C_i^{(\mathrm{II})} = g_{\psi i}(C_i^{(\mathrm{I})}) = g_{\psi i}[g_{wi}(C_i)] = g_{\psi wi}(C_i)$
$w(x) = \sum_{i=1}^{6} g_{wi}(C_i) f_i \quad \psi(x) = \sum_{i=1}^{6} g_{\psi wi}(C_i) f_i$
(g_{wi}, $g\psi_{wi}$: functions depending on the solution case)

Step 3: *Calculation of constants C_I*
From the boundary conditions (Fig. 2):

$$(x=0) \Rightarrow \begin{cases} \psi = 0 \\ w_k' = 0 \\ EI(\psi'') = \frac{P}{2} \end{cases} \Rightarrow \begin{cases} C_2 \\ C_5 \ (x \to \infty) \Rightarrow w_k = 0 \Rightarrow \\ C_6 \end{cases} \begin{cases} C_1 \\ C_3 \\ C_4 \end{cases}$$

Step 4: *Calculation of stresses $M(x), V(x), V_G(x)$*
$M(x) = (EI)\psi'(x) \quad V(x) = (EI)\psi''(x) \quad V_G(x) = G w_k'(x)$
(See Appendix B)

4. Beams of infinite length on three-parameter elastic foundation

A first application of Eqs. (8a), (8b) and (9) is the solution of the problem of beams of infinite length. The usefulness of the solution of the former problem lies in the fact that the comparison between various elastic foundation models can be carried out without the complications (i.e., singularities) which occur at the ends of finite length beams. Furthermore, the solution for beams of infinite length is, in many cases, an acceptable approximation for the solution of the respective problem of long beams.

In what follows, a brief account of the calculations made in the case of Timoshenko beams under concentrated vertical load will be given. The method starts with the solutions (Eq. (12)) of Eqs. (8a) and (8b). The sequence of the necessary steps is outlined in Table 1. The total number of the unknown constants of these equations is 18, since each of the three unknown strains w, w_k and ψ demands the estimation of six integration constants. The boundary conditions produced by the application of the principle of stationary total potential energy (Eq. (5b), in which boundaries $x = 0$ and L are substituted by boundaries $x \to \pm\infty$), are shown in Fig. 2. The number of boundary conditions is smaller than the number of the unknown constants. However, the 18 constants are interrelated according to Eqs. (7c) and (9).

More specifically, it can be easily shown (Morfidis, 2003) that the integration constants can be expressed as a function of only 6 constants (C_1 to C_6). It is therefore clear that, the symmetry of the problem also being considered, the boundary conditions shown in Fig. 2 are sufficient for the calculation of the constants C_i.

Having calculated C_i, the respective stresses and strains in the beam can be determined. The analytical expressions of the stresses and strains for solution case 1 (Appendix A) are given in Appendix B.

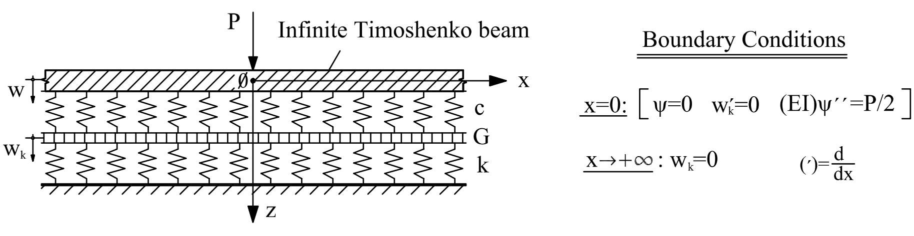


Fig. 2. Mechanical model and boundary conditions for a Timoshenko beam of infinite length on three-parameter elastic foundation.

5. Beams of finite length on three-parameter elastic foundation

The solution of the problem of beams of finite length is based on the simultaneous solution of Eqs. (6a), (6b), (8a), (8b) and (9), which requires the determination of a total of 22 integration constants. However, the integration constants that correspond to Eqs. (8a), (8b) and (9) of region II are interrelated through Eqs. (7c) and (9) as noted in the preceding section. As a result, the number of the required constants is 10 and the boundary conditions of Fig. 1 are sufficient for their determination. The application of these boundary conditions leads to a system of 10 algebraic equations, through which the unknown constants can be calculated. Having determined the 10 integration constants, it is possible to calculate the strains (w, w_k, w_c and ψ) and the respective stresses (M, V, V_G—see Fig. 1).

6. Numerical examples

In this section, two simple yet typical problems are solved: (A) The problem of beams of infinite length under a concentrated vertical force, and (B) the corresponding problem of beams of finite length under the same type of loading. The main objectives of these two examples are:

a. The comparative evaluation of one, two and three-parameter models. It is for this purpose that a comparison is made between the results of these models and those obtained from the use of 2-D finite element models. Apart from the Kerr model, the other models under evaluation are (Fig. 3): (i) the Winkler model (one-parameter model), (ii) the Pasternak model (two-parameter model, in which the influence of the soil on either side of the beam is not considered), and (iii) the Vlasov model (two-parameter model, in which the influence of the soil on either side of the beam is, in fact, considered).
b. The application of a newly developed analytical method for a meaningful numerical estimation of values for the soil parameters of the Kerr model. The proposed method is based on modified Vlasov model (see Section 7).

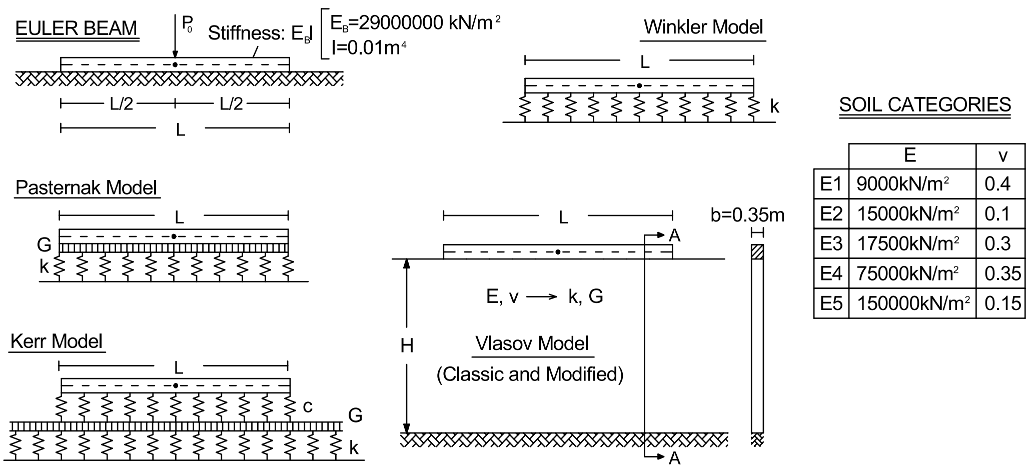


Fig. 3. Beams of finite length under concentrated vertical force—models and soil categories used in the numerical examples.

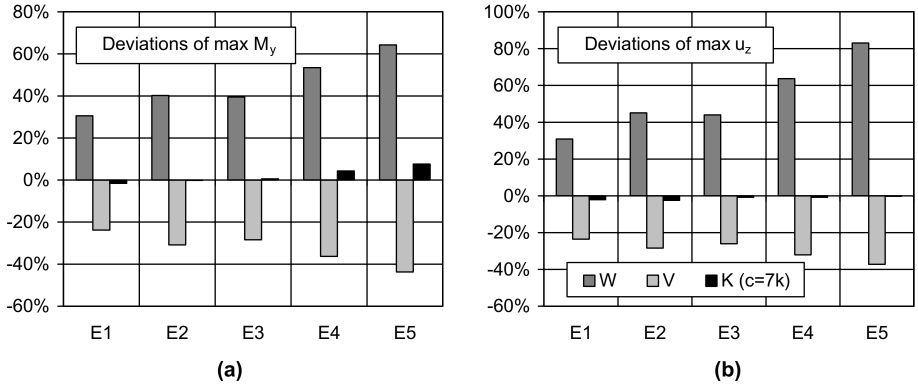
It must be underlined that the choice of 2-D finite element models as reference models (‘reference solution’) is based on the fact that they yield the most precise results within the framework of the fundamental assumptions, on the basis of which the soil-beam interaction equations of the approximating mechanical models under analysis are formulated. These fundamental assumptions are: (a) the assumption of the linear elastic behavior of the beam and the subgrade, and, (b) the assumption of conditions of bilateral (in contrast with unilateral) contact between them. It is also worth mentioning that the solution using 2-D finite elements is the most precise one possible in the case of plane stress problems. For this purpose it has been assumed that plane stress conditions are valid and the width of the elastic subgrade is $b = 0.35$ m (Fig. 3).

For the numerical investigation of the two problems, (A) and (B), two special algorithms were programmed using Fortran 90/95. The first algorithm refers to the solution for one and two-parameter models, and calculates the parameters k and G using the modified Vlasov model. The second algorithm relates to the three-parameter Kerr model. This algorithm leads to an estimation of the values of the third parameter (parameter c) of the Kerr model, while making use of the values of the parameters k and G that have been obtained using the first algorithm. (For a detailed description of the underlying methodology see Section 7). The finite element program SAP2000 (2000) was employed for the solutions using 2D finite element models (see also Morfidis and Avramidis, 2002).

With respect to the numerical values of the elastic foundation constants, five categories of soil were considered, ranging from “very soft” to “hard” (Fig. 3). Since the depth of the elastic subgrade was set to 60 m, it can be practically characterized as one of infinite depth. The items under investigation were the vertical displacement (max u_z) and the bending moment (max M_y) at the point of application of the external force, where their largest values are encountered. Furthermore, the beam deformation curves are drawn in order to serve as an indicator of the mechanical performance of the various soil models.

6.1. Beams of infinite length under concentrated vertical force

The comparison of results derived from the analysis of beams of infinite length (Fig. 2) constitutes a most useful criterion in the evaluation of soil models, as they are not influenced (in contrast to finite length beams) by the boundary conditions at the ends of the beams or by their length. Fig. 4 shows a comparison of the three models under examination (Winkler, Vlasov and Kerr models) as to the deviations from the reference results (i.e. the 2-D finite element model results) of both the maximum bending moments M_y



$$Deviation\ (\%) = \left[\frac{(MMV)}{(RMV)} - 1\right] \bullet 100$$

MMV = Mechanical Model Value (Fig.3)

RMV = Reference Model Value (2 - D FEM - Model)

Fig. 4. Deviations of (a) the maximum bending moments M_y and (b) the maximum vertical displacements u_z of the Winkler (W), Vlasov (V) and Kerr (K) models from the respective values obtained from the reference 2-D finite element solution.

(Fig. 4a) and the maximum vertical displacements u_z (Fig. 4b). From Figs. 4a and b, three conclusions may be drawn:

a. The superiority of the Kerr model over the other models is evident in the comparison of the maximum bending moments as well as the maximum vertical displacements. In fact, when $c = 7k$ ($n_{ck} = 7$, see Section 7), this model displays deviations that do not exceed 7.5% in any case.
b. In general, the Winkler model produces higher values of bending moments and displacements than the reference solution. More specifically, the deviations of the bending moments are more significant in the case of "hard" soils (E4 and E5), and may amount up to 65%. The deviations of the vertical displacements are of the same order of magnitude as the deviations of the bending moments.
c. The values of the bending moments and vertical displacements obtained from the Vlasov model are, on the whole, smaller than the respective reference values. At the same time, the absolute values of their deviations are smaller than the respective deviations of the Winkler model. As shown in Fig. 4a, the deviations of the bending moments in the case of the "hard" soils E4 and E5 do not rise above 45%. Again, as in the case of the Winkler model, the deviations of the displacements are of the same order of magnitude as the deviations of the bending moments.

The superiority of the Kerr model against all other investigated models can be easily concluded from Fig. 5, where the beam deformation curves are shown (Soil category E3 is not included because its results almost coincide with the results for soil category E2). The following remarks can be made:

a. The deformation curves of the Kerr model are a very good approximation of the 2D-FEM deformation curves along the whole length of the beam. (For $x > 50$ m they are practically identical).

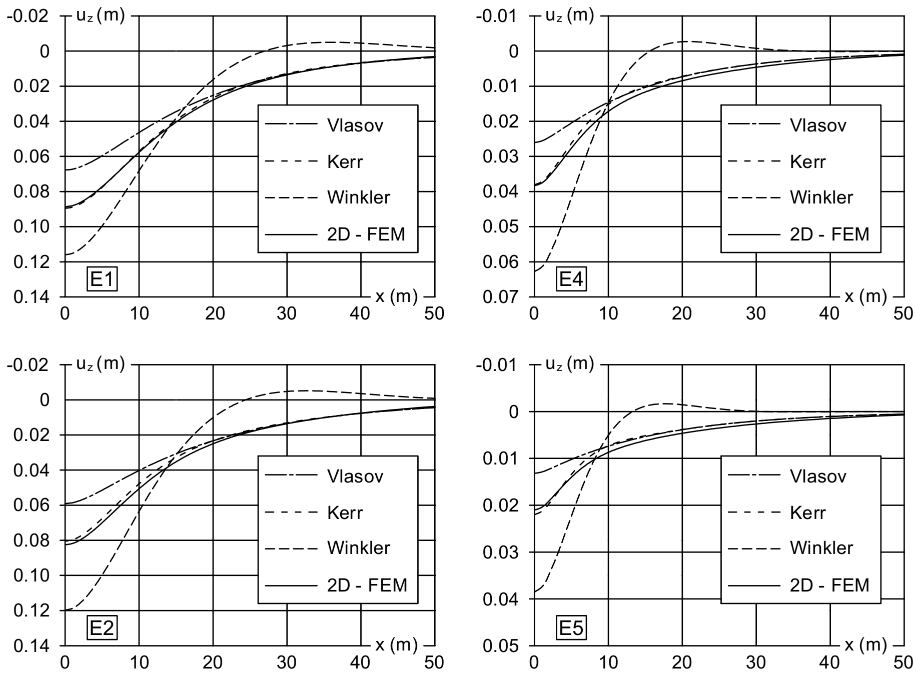


Fig. 5. Deformation curves of an infinite length beam under concentrated load.

b. The deformation curves of the Vlasov model show major divergences for $x < 10$ m in case of ''hard'' soils (soil categories E4 and E5) and for $x < 15$ m in case of ''softer'' soils (soil categories E1 and E2).
c. The Winkler model deformation curves significantly diverge from the 2D-FEM deformation curves along the whole length of the beam. Furthermore, they also qualitatively differ from the curves of all the other soil models.

6.2. Beams of finite length under concentrated vertical force

The second example concerns with the problem of a finite length beam. The same type of loading as in the previous case of infinite length beam is considered. This very simple type of loading has been choosen as a first step towards the assessment of the comparative performance of the investigated soil models. Solutions for other types of loading, which may be useful for a detailed and in-depth validation of the mechanical behavior of the soil models, will be given in a following paper. The methodology used for the estimation of the soil parameters' values is described in the next section (Section 7).

Fig. 6 shows comparisons between the values obtained from the use of the models of Fig. 3 and the values of the reference solution (2D-FEM solution), with respect to M_y and u_z. These results refer to beams for which the factors λ (see Eq. (16)) are equal to 0.94 ($L = 6$m—soil E1) or 1.10 ($L = 10$ m—soil E4). These values of λ are at the limit between regions 2 and 3 (see Table 2 in Section 7), where the three-parameter

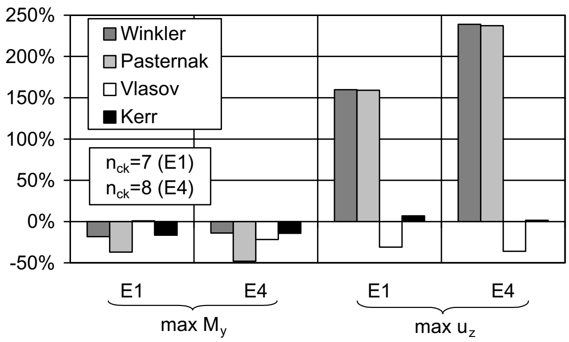


Fig. 6. Deviations (in %) of the maximum bending moments M_y and the maximum vertical displacements u_z of the four foundation models under investigation from the respective reference (2D-FEM model) solution values.

Table 2
Relation of the optimal relation factors n_{ck} to the relative stiffness index λ

Region	Limits of the value of λ		Comment
1	E1	$\lambda \leqslant 0.9$–0.95	Total convergence at the reference solution is accomplished. The factor n_{ck} is related to λ through polynomial functions (Fig. 9).
	E4	$\lambda \leqslant 0.7$	
2	E1	0.9–$0.95 \leqslant \lambda \leqslant 1.1$–$1.2$	Total convergence is not accomplished. For values of $n_{ck} \approx 10^4$, the deviations are smaller than 10%.
	E4	$0.7 \leqslant \lambda \leqslant 0.9$–$0.95$	
3	E1	1.1–$1.2 \leqslant \lambda \leqslant 1.65$–$1.7$	Total convergence is not accomplished. When $n_{ck} \approx 7$ (E1) or $n_{ck} \approx 9$ (E4), the deviations are smaller than 10–15%.
	E4	0.9–$0.95 \leqslant \lambda \leqslant 1.6$–$1.6$	
4	E1	$\lambda \geqslant 1.65$–1.7	Total convergence is accomplished. For $n_{ck} \approx$ 7–9 (E1) and $n_{ck} \approx$ 9–13 (E4), the deviations are smaller than 4%.
	E4	$\lambda \geqslant 1.55$–1.6	

model appears to be somewhat ineffective in terms of the convergence of the values of the bending moments.

On the basis of Fig. 6, the following remarks can be made:

- As far as the maximum vertical displacements are concerned, the superiority of the three-parameter Kerr model is obvious. This particular model displays deviations from the reference values that are no higher than 6%. The results obtained from the application of the Vlasov model are satisfactory as well (deviations up to 30–35%). In contrast, the results of the Winkler and Pasternak models lead to deviations that exceed 150% and must, therefore, be rejected. From all this, it may be inferred that the consideration of the deformations of the soil on either side of the beams constitutes a fundamental parameter for the correct calculation of the vertical displacements of the foundation beams. Only two of the ‘‘mechanical’’ models under examination, the Vlasov and Kerr models, offer this potential.
- As to the maximum bending moments, the deviations are substantially smaller than those of the maximum vertical displacements. Three additional assumptions may be made:
  a. The Vlasov and Kerr models (especially the first) yield the most precise results. This is particularly evident in the case of the soil E1, in which the Winkler and Pasternak models prove to be inferior to the other two models, though not considerably. More specifically, the Kerr and Winkler models appear to be of equal merit.
  b. In the case of the soil E4 (hard soil), the Kerr and Winkler models are equally efficient (with deviations of the order of –14%). The Vlasov model is slightly inferior (deviation equal to –21%), while the Pasternak model is clearly the least reliable in the group (deviation equal to –48%).
  c. The consideration of the influence of the soil on either side of the beams is not of the same significance as in the case of the convergence of the vertical displacements.

In Fig. 7 the beam deformation curves for soil categories E1 and E4 are shown. These results further underpin the previous conclusions referring to the vertical displacements drawn from Fig. 6.

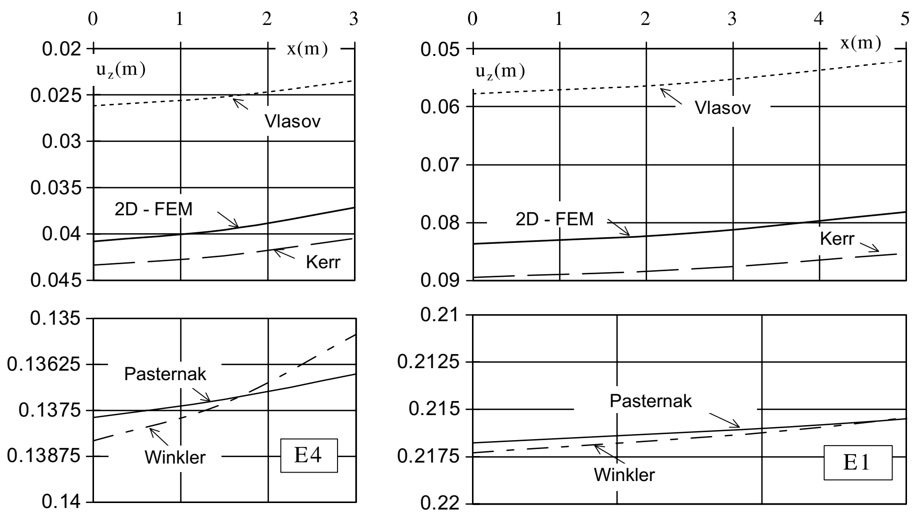


Fig. 7. Deformation curves of a finite length beam under concentrated load at the centre.

7. Estimation of the values of the soil parameters

The problem of specifying realistic values of the soil parameters involved in two- and three-parameter models has already been addressed in the bibliography (e.g. Selvadurai, 1979; Kerr, 1985). The lack of appropriate laboratory tests or in situ measurement methodologies, through which it would be possible to overcome this problem, has also been acknowledged (Selvadurai, 1979). Concerning the simpler two-parameter models, the sole method of defining the two soil parameters reported in the bibliography is the analytical method of Vallabhan and Das (1988, 1991a,b), which is founded on a quite different formulation of the equations of the two-parameter Vlasov model (modified Vlasov model). The solution using the modified Vlasov model demands the programming of an iterative algorithm, thanks to which the simultaneous calculation of the two soil parameters is made possible.

With regard to three-parameter models, it is possible to relate their parameters with the modulus of elasticity E_s, the shear modulus G_s, and the depth H of the elastic subgrade, through Reisner's relations (Reissner, 1967):

$$c_1 = E_s/H \quad c_2 = (G_sH)/(3) \quad c_3 = (G_sH^2)/(12E_s) \tag{13}$$

In addition, a relation between the parameters of the Reisner model and the respective parameters of the Kerr model can be derived, if (Morfidis, 2003):

$$c = 3k \quad k = (4E_s)/(3H) \quad G = (4HG_s)/9 \tag{14}$$

In the present paper, a newly developed method for a meaningful numerical estimation of the values of the three parameters is proposed. According to this method, the modified Vlasov model will be employed in order to estimate two of the three soil parameters of the Kerr-type three-parameter model. It must be emphasized that the parameter k (constant of the lower spring layer) and the parameter G (constant of the shear layer) of the Kerr model correspond to the two respective parameters of the Vlasov model. It is reminded that the two soil parameters of the Vlasov model are calculated by taking into account the strain energy associated to the normal and shear stresses inside the elastic foundation (Vlasov and Leontiev, 1966). Therefore, the values of k and G that are obtained from the application of the modified Vlasov model can be identified as the values of the respective parameters of the Kerr model. With regard to the third parameter c of the Kerr model (constant of the upper spring layer), it is assumed that it is related to the parameter k in the following way:

$$c = n_{ck}k \tag{15}$$

The factor n_{ck} is a "relation factor" for the parameters c and k and expresses the relative axial stiffness of the upper and lower spring layers. The relation of k and c through Eq. (15) is well-defined, since the parameters k and c represent constants of vertical springs. In the following, an investigation is carried out concerning the possible existence of *optimal* relation factors n_{ck}, through which the greatest possible convergence between the results of the Kerr model and those obtained from the use of 2-D finite element models (which are regarded as "reference solution") is achieved.

7.1. Beams of infinite length under concentrated vertical force

Fig. 8 illustrates the results of the investigation of the existence of optimal relation factors n_{ck}, which was based on the criterion of the convergence of the values of max u_z and max M_y at the corresponding values of the reference solution (i.e., the 2-D finite element model solution).

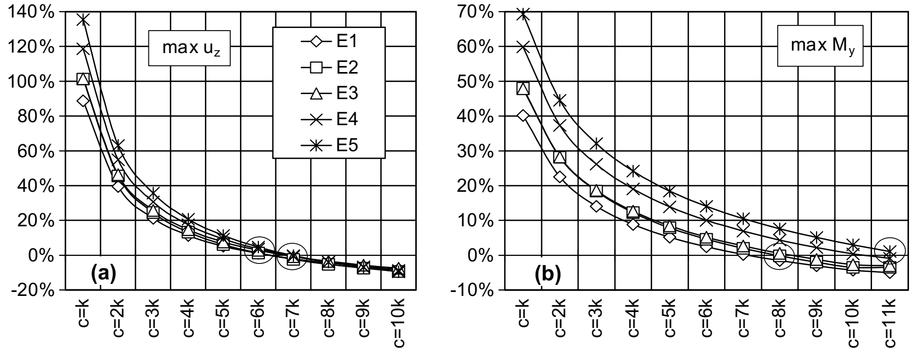


Fig. 8. Investigation of the optimal relation factors n_{ck} with respect to the soil class (E1–E5) and the constant ratio of the spring c/k. Deviations (in %) from the reference values for (a) max u_z, and (b) for max M_y.

From Fig. 8, the following conclusions emerge:

a. For the "soft" soils E1, E2 and E3—as far as the maximum bending moment M_y is concerned—a considerably close convergence to the reference results is achieved (small deviations of the order of 0.5–2.5%), when the factor n_{ck} acquires values between 7 and 8 ($c = 7-8k$). The optimal value of n_{ck}, for which the convergence of max u_z at the respective reference value is achieved, ranges between 6 and 7 ($c = 6$–$7k$). More particularly, the deviations calculated for these values of the spring constant c do not exceed 2.5%. On the basis of the above remarks, it may be suggested that, in the case of soils with a low modulus of elasticity, the optimal value of n_{ck} is equal to 7 ($c = 7k$). This value produces an optimal simultaneous convergence of both bending moments and vertical displacements at the respective reference values.
b. For the "hard" soils E4 and E5, the optimal convergence of the maximum bending moment max M_y is achieved when the value of n_{ck} ranges between 10 and 11 ($c = 10$–$11k$). For these values, the deviations of the bending moments do not exceed 2.5%. With respect to the optimal convergence of the vertical displacements, this is achieved for $n_{ck} = 7$ (deviations smaller than 1%). For "hard" soils, as opposed to "soft" soils, no single value of n_{ck}, which would result to an optimal convergence of both max u_z and max M_y, can be suggested.

7.2. Beams of finite length under concentrated vertical force

As part of the investigation of the solution for finite length beams, a parametric analysis was carried out, in order to determine the dependence of the optimal relation factors n_{ck} on the index which describes the relative stiffness of the soil-beam system. The definition of this index required the use of Vlasov's relation (Vlasov and Leontiev, 1966):

$$\lambda = \sqrt[3]{\frac{(1 - v_s)G_s b L^3}{8EI(1 - 2v_s)}} \tag{16}$$

where G_s and v_s are the shear modulus and Poison's ratio for the soil medium respectively, EI is the stiffness of the beam, L is its length and b the width of its cross-section.

The cases examined were those of the soils $E1$ and $E4$, i.e., of soils with a low and relatively high modulus of elasticity respectively. As in the study of beams of infinite length, the quantities compared are the maximum vertical displacement u_z and the maximum bending moment M_y. The parametric investigation led to the assumption that, with regard to vertical displacements, the Kerr model achieves the optimal convergence at the reference values when $n_{ck} = 7$ (for "soft" soils) or $n_{ck} = 8$ (for "hard" soils). These values produce deviations from the reference solutions that do not exceed 6%. However, even if the value of n_{ck} is set to 7 regardless of the soil type (whether "hard" or "soft"), the deviations are in no case higher than 10%. (At this point, it should be clarified that the modified Vlasov model was employed for the calculation of the parameters k and G of the Kerr model, as was pointed out in Section 7). On the contrary, in the case of maximum bending moments M_y, it is not possible to determine a single value of n_{ck} for which the optimal performance of the Kerr model is attained. The parametric analyses proved that the optimal values of n_{ck} depend on the elastic features of the beam and the soil, as well as on the length of the beam. As a result, the most appropriate way of modifying the value of n_{ck} is to relate it to the index λ of the relative stiffness of the soil-beam system (Eq. (16)). The form of the relation between the parameters n_{ck} and λ is presented in the following table.

From Table 2, the following may be concluded:

- For small values of λ (region 1), the determination of the optimal factors n_{ck} demands non-linear regression analyses. A number of such analyses established the fact that the curves n_{ck}–λ can be grouped according to the modulus of elasticity of the soil (i.e., "soft" or "hard" soil). In Fig. 6, the curves n_{ck}–λ are given for the soils E1 (soft soil) and E4 (hard soil). This figure shows the curves n_{ck}–λ which correspond to the results of the analyses (and are, in essence, point sequences), as well as the curves (bold lines) which correspond to the polynomial relations derived from the non-linear regression analyses. From Fig. 9, it becomes clear that the curves n_{ck}–λ exhibit a high degree of correlation ($R^2 \approx 1$), i.e., there exists a strong interdependence of the parameters n_{ck} and λ.
- For mean values of λ (region 2), the Kerr model, in conjunction with the method used in order to determine its parameters, does not achieve a total convergence of the values of the bending moments at the reference values. Nevertheless, when n_{ck} acquires very large values ($n_{ck} \approx 10^4$), the deviations do not rise above 10%.
- For large values of λ (regions 3 and 4), convergence is attained when the values of n_{ck} are small. In region 4, there is absolute coincidence between the values of λ and the reference values, a fact not observed in region 3. At this point, it is worth noting that if $\lambda > 1.65$, the optimal factors n_{ck} are practically identical with the optimal factors used for the convergence of the maximum bending moments of beams of infinite length.

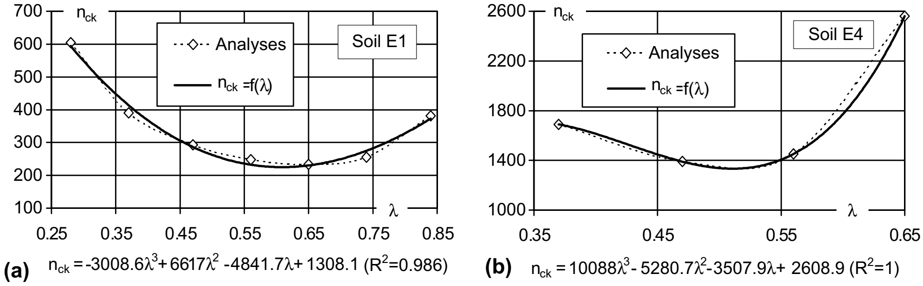


Fig. 9. Relation of the parameters n_{ck} and λ to polynomial functions for small values of λ.

8. Summary and conclusions

The problem of bending of Timoshenko beams on a Kerr-type, three-parameter elastic foundation was presented and analytically solved. The problem was stated by formulating the differential equations of equilibrium, as well as the necessary boundary conditions by using the principle of stationary potential energy. A first conclusion reached is that there are only two possible forms of solution if the mechanical parameters of the beams and of the soil acquire realistic values, i.e., values that fall within the usual range for realistic beams and soil types. In this case, the solutions do not exhibit any peculiarity, as are uniquely defined.

A rational procedure of numerically estimating the soil parameters involved in the two and three parameter models was also presented. The proposed procedure comprises the simultaneous application of the two-parameter, modified Vlasov model and the three-parameter, Kerr model and has been shown to be quite efficient, thus bypassing the necessity of experimental determination.

Furthermore, the numerical efficiency of the three-parameter model formulation was verified by means of simple numerical examples and through comparison with one- and two-parameter models. The solutions of problems of beams of infinite and finite length under a concentrated vertical load formed the basis for this comparison, which led to the following additional conclusions:

(a) It is possible to assign specific values to the three parameters of the Kerr model, from which results are recovered that agree to a high degree with those obtained from use of 2-D finite element models. The specific values of these parameters are derived during application of the aforementioned procedure.
(b) In problems concerning beams of finite length, the values of the above parameters depend strongly on the relative stiffness of the soil-beam system.
(c) The model which produces the best results is the Vlasov model, especially in regards to the vertical displacements of infinite length beams.
(d) In case of beams of finite length, the consideration of the soil on either side of the beams is of paramount importance if acceptable results for the vertical displacements are to be achieved. This consideration was found to be less significant for the calculation of the bending moments.

Appendix A. Solution of the homogenous part of the differential equations (8a) and (8b)

The homogenous part of Eqs. (8a) and (8b) has the following form:

$$\frac{d^6y}{dx^6}+J_1\frac{d^4y}{dx^4}+J_2\frac{d^2y}{dx^2}+J_3y=0 \tag{A.1}$$

$$J_1=-\left(\frac{k+c}{G}+\frac{c}{\Phi}\right)\quad J_2=\frac{c}{\Phi}\left(\frac{\Phi}{EI}+\frac{k}{G}\right)\quad J_3=-\frac{kc}{EIG} \tag{A.2}$$

(where y may represent of the functions w, w_k, ψ).

The auxiliary equation of (A.1) is:

$$r^6+J_1r^4+J_2r^2+J_3=0 \tag{A.3}$$

Considering that $r^2=\mu$, the following relation is obtained:

$$\mu^3+J_1\mu^2+J_2\mu+J_3=0 \tag{A.4}$$

With the additional consideration that $\mu = t - (J_1/3)$, (A.4) becomes:

$$t^3 + (3\alpha)t + 2\beta = 0 \tag{A.5}$$

where

$$\alpha = (1/3)[-(J_1^2/3) + J_2] \text{ and } \beta = (1/2)[(2J_1^3/27) - (J_1J_2/3) + J_3] \tag{A.6}$$

Depending on the sign of the quantity $\Delta = \alpha^3 + \beta^2$, the solution types of (A.5) are divided into the following general solution categories:

Category A: $\Delta = \alpha^3 + \beta^2 > 0$: in this case, the root t_1 of (A.5) is a real number, while t_2 and t_3 are complex numbers. As a result, the root μ_1 of (A.3) is a real number and μ_2, μ_3 are complex numbers. In this category, the sign of μ is significant, since it leads to two subcategories of solution.

Category B: $\Delta = \alpha^3 + \beta^2 < 0$: here, t_1, t_2, t_3, and thus $\mu_1\mu_2$ and μ_3 are real numbers. In this case, the general solution type of (A.1) is dependent on the sign of μ_1, μ_2 and μ_3. Therefore, in the general case, it is possible to encounter forms of solution that are as many as the combinations of the signs of the roots μ_1, μ_2 and μ_3. Consequently, if the possibility that some of the roots μ_1, μ_2, μ_3 may be equal is excluded, the possible solution cases of category B amount to 8.

Category C: $\Delta = \alpha^3 + \beta^2 = 0$: in this case yet again, the roots t_1, t_2, t_3, and thus the roots μ_1, μ_2, μ_3 are real numbers. In practice, however, this category of solution is extremely unlikely to come up, as this would require certain combinations of the values of the three soil parameters.

On the basis of the remarks made above, and assuming that a solution belonging to category C is almost impossible to occur, it is determined that the possible solution cases of the homogenous equations of (8a), (8b) are 10.

These possible solution cases may be further investigated using one of the three relations through which the roots of (A.4) are associated (Viéta's formulas: see i.e. Borwein and Erdélyi, 1995):

$$\mu_1\mu_2\mu_3 = -J_3 \tag{A.7}$$

By using (A.7) with category *A*, it is concluded that the sign of μ_1 is the opposite of that of the quantity J_3 (provided that the product of the complex conjugate numbers μ_2 and μ_3 is always a real positive number). Nevertheless, the quantity J_3 is always negative and thus the root μ_1 is always positive. Therefore, a possible solution case of the homogenous equations of (8a) and (8b) is:

Case 1	f_1	f_2	f_3	f_4	f_5	f_6
	$e^{(R_1)x}$	$e^{-(R_1)x}$	$e^{(R)x}\cos(Qx)$	$e^{(R)x}\sin(Qx)$	$e^{-(R)x}\cos(Qx)$	$e^{-(R)x}\sin(Qx)$

$$R_1 = \sqrt{\sqrt[3]{-\beta+\sqrt{\Delta}} + \sqrt[3]{-\beta-\sqrt{\Delta}} - (J_1/3)} \qquad R = \sqrt{(\sqrt{m^2+n^2}+m)/2} \qquad Q = \sqrt{(\sqrt{m^2+n^2}-m)/2}$$

$$m = -(1/2)\left[\sqrt[3]{-\beta+\sqrt{\Delta}} + \sqrt[3]{-\beta-\sqrt{\Delta}} + (2J_1/3)\right] \qquad n = (\sqrt{3}/2)\left[\sqrt[3]{-\beta+\sqrt{\Delta}} - \sqrt[3]{-\beta-\sqrt{\Delta}}\right]$$

In order to investigate the possible subcategories of category B, the signs of the real roots of Eq. (A.4) should be checked. It is for this purpose that the function $f(\mu) = \mu^3 + J_1\mu^2 + J_2\mu + J_3$ is considered. The signs of the coefficients J_1, J_2 and J_3 is always $J_1 < 0$, $J_2 > 0$, $J_3 < 0$. It can be proved that, for the aforementioned signs of J_1, J_2 and J_3, the first derivative of $f(\mu)$ has two positive roots (i.e., μ_1' and μ_2'), for which $\mu_1' < \mu_2'$, $f(\mu_1') = 2(\sqrt{-\alpha^3} + \beta) > 0$, and $f(\mu_2') = 2(-\sqrt{-\alpha^3} + \beta) < 0$. In addition, $f(0) = J_3 < 0$. As a result, $f(0)f(\mu_1') < 0$ and $f(\mu_1')f(\mu_2') < 0$, indicating that Eq. (A.4) has one root in the interval $(0, \mu_1')$ and another in (μ_1', μ_2'). However, given that $\mu_1' > 0$ and $\mu_2' > 0$, (A.4) has two positive roots. At the same time, from Eq. (A.7), it is obtained that the third root must be positive and, as a consequence $\mu_1 > 0$, $\mu_2 > 0$ and $\mu_3 > 0$. The second possible solution subcategory of the homogenous equations of (8a) and (8b) is thus expressed in the following way:

Case 2	f_1	f_2	f_3	f_4	f_5	f_6
	$e^{(R)x}$	$e^{-(R)x}$	$e^{(R_1)x}$	$e^{(R_2)x}$	$e^{-(R_1)x}$	$e^{-(R_2)x}$
	$R = \sqrt{2\sqrt{-\alpha}\cos\left(\frac{\phi}{3}\right) - \frac{J_1}{3}}$		$R_1 = \sqrt{2\sqrt{-\alpha}\cos\left(\frac{\phi+2\pi}{3}\right) - \frac{J_1}{3}}$		$R_2 = \sqrt{2\sqrt{-\alpha}\cos\left(\frac{\phi+4\pi}{3}\right) - \frac{J_1}{3}}$	

where $\phi = \cos^{-1}(-\beta/\sqrt{-\alpha^3})$.

Appendix B

Stresses and strains for beams of infinite length on three-parameter elastic foundation (Solution Case 1):

$$w(x) = \frac{P}{D_k}\{A_1(A_5Q - A_6R)e^{-R_1x} + [-L_1\cos(Qx) + L_2\sin(Qx)]e^{-Rx}\}$$

$$L_1 = A_3(A_4Q - A_6R_1) + A_2(A_4R - A_5R_1), \quad L_2 = A_2(A_4Q - A_6R_1) - A_3(A_4R - A_5R_1)$$

$$w_k(x) = \frac{P}{D_k}\{(A_5Q - A_6R)e^{-R_1x} - [(A_4Q - A_6R_1)\cos(Qx) + (A_4R - A_5R_1)\sin(Qx)]e^{-Rx}\}$$

$$\psi(x) = \frac{P}{D_k}\{-A_4(A_5Q - A_6R)e^{-R_1x} + [L_3\cos(Qx) + L_4\sin(Qx)]e^{-Rx}\}$$

$$L_3 = -A_6(A_4R - A_5R_1) + A_5(A_4Q - A_6R_1), \quad L_4 = A_5(A_4R - A_5R_1) + A_6(A_4Q - A_6R_1)$$

$$M(x) = -\frac{EIP}{D_k}\{-A_4R_1(A_5Q - A_6R)e^{-R_1x} - [(QL_4 - RL_3)\cos(Qx) - (QL_3 + RL_4)\sin(Qx)]e^{-Rx}\}$$

$$V(x) = -\frac{EIP}{D_k}\left\{A_4R_1^2(A_5Q - A_6R)e^{-R_1x} - \begin{bmatrix}[L_3(R^2 - Q^2) - 2RQL_4]\cos(Qx) + \\ +[L_4(R^2 - Q^2) + 2RQL_3]\sin(Qx)\end{bmatrix}e^{-Rx}\right\}$$

$$V_G(x) = \frac{GP}{D_k}\{-R_1(A_5Q - A_6R)e^{-R_1x} + [S_1\cos(Qx) + S_2\sin(Qx)]e^{-Rx}\}$$

$$S_1 = R(A_4Q - A_6R_1) - Q(A_4R - A_5R_1), \quad S_2 = R(A_4R - A_5R_1) + Q(A_4Q - A_6R_1)$$

$$D_k = 2EI\left[R_1(A_5b_2 - A_6b_1) + R_1^2A_4(A_6R - A_5Q) + A_4(b_1Q - b_2R)\right]$$

$$b_1 = (R^2 - Q^2)A_5 - (2RQ)A_6, \quad b_2 = (R^2 - Q^2)A_6 + (2RQ)A_5$$

$$A_1 = \left(1 + \frac{k}{c}\right) - \frac{G}{c} R_1^2, \quad A_2 = 2RQ\left(\frac{G}{c}\right), \quad A_3 = \left(1 + \frac{k}{c}\right) - \frac{G}{c}(R^2 - Q^2), \quad \begin{aligned} \alpha_1 &= A_3 R + A_2 Q \\ \alpha_2 &= A_2 R - A_3 Q \end{aligned}$$

$$A_4 = \left[\frac{\Phi R_1}{-EIR_1^2 + \Phi}\right] A_1, \quad A_5 = \left(\frac{\alpha_1 \beta_1 + \alpha_2 \beta_2}{\beta_1^2 + \beta_2^2}\right)\Phi, \quad A_6 = \left(\frac{\alpha_1 \beta_2 - \alpha_2 \beta_1}{\beta_1^2 + \beta_2^2}\right)\Phi, \quad \begin{aligned} \beta_1 &= -EI(R^2 - Q^2) + \Phi \\ \beta_2 &= 2(EI)RQ \end{aligned}$$

References

Borwein, P., Erdélyi, T., 1995. Polynomials and Polynomial Inequalities. Springer-Verlag, New York.

Filonenko-Borodich, 1940. Some approximate theories of elastic foundation. Uchenyie Zapiski Moskovskogo Gosudarstvennogo Universiteta, Mekhanica 46, 3–18 [in Russian].

Hetenyi, M., 1946. Beams on elastic foundation. Scientific Series, vol. XVI. Ann Arbor: The University of Michigan Press, University of Michigan Studies.

Hetenyi, M., 1950. A general solution for the bending of beams on an elastic foundation of arbitrary continuity. Journal of Applied Physics 21, 55–58.

Horvath, J.S., 1983a. New subgrade model applied to Mat foundations. Journal of Geotechnical Engineering 109 (12), 1567–1587.

Horvath, J.S., 1983b. Modulus of subgrade reaction: new perspective. Journal of Geotechnical Engineering 109 (12), 1591–1596.

Horvath, J.S., 1993. Beam-Column-Analogy Model for soil—structure interaction analysis. Journal of Geotechnical Engineering 119 (2), 358–364.

Kerr, A.D., 1965. A study of a new foundation model. Acta Mechanica 1/2, 135–147.

Kerr, A.D., 1985. On the determination of foundation model parameters. Journal of Geotechnical Engineering 111 (11), 1334–1340.

Morfidis, K., Avramidis, I.E., 2002. Formulation of a generalized beam element on a two-parameter elastic foundation with semi-rigid connections and rigid offsets. Computers and Structures 80, 1919–1934.

Morfidis, K., 2003. Research and development of methods for the modeling of foundation structural elements and soil. PhD dissertation, Department of Civil Engineering, Aristotle University of Thessaloniki.

Pasternak, P.L., 1954. On a new method of analysis of an elastic foundation by means of two-constants. Moscow, USSR: Gosudarstvennoe Izdatelstvo Literaturi po Stroitelstvu I Arkhitekture [in Russian].

Reissner, E., 1967. Note on the formulation of the problem of the plate on an elastic foundation. Acta Mechanica 4, 88–91.

SAP Non-Linear Version 7.42., 2000. Integrated finite element analysis and design of structures. Berkeley, California, USA: Computers and Structures, Inc.

Selvadurai, A.P.S., 1979. Elastic Analysis of Soil–foundation Interaction. Elsevier, Amsterdam.

Vlasov, V.Z., Leontiev, U.N., 1966. Beams, plates, and shells on elastic foundation. Jerusalem: Israel Program for Scientific Translations [translated from Russian].

Vallabhan, C.V.G., Das, Y.C., 1988. Parametric study of beams on elastic foundations. Journal of Engineering Mechanics 114 (12), 2072–2082.

Vallabhan, C.V.G., Das, Y.C., 1991a. Modified vlasov model for beams on elastic foundations. Journal of Geotechnical Engineering 117 (6), 956–966.

Vallabhan, C.V.G., Das, Y.C., 1991b. A refined model for beams on elastic foundations. International Journal of Solids and Structures 27 (5), 629–637.

Winkler, E., 1867. Die Lehre Von Der Elastizitat Und Festigkeit. Dominicus, Prague.